348

PROVE that, if $0 < a < \beta$,

$$\int_a^\beta \log \frac{\beta - x}{x - a} \frac{dx}{x} = \tfrac{1}{2} \left[\log \frac{\beta}{a} \right]^{2*}.$$

[*Frank Morley.*]

SOLUTION.

The integration is best effected by use of the following theorem, or lemma·

$$\int_a^\beta \varphi \left[\frac{x}{\beta} \right] \frac{dx}{x} = \int_a^\beta \varphi \left[\frac{a}{x} \right] \frac{dx}{x},$$

where $\varphi(z)$ is any finite and continuous function of z; and β and a have any fixed values, provided when one is zero the other must be infinite.

To prove this, let $\dfrac{x}{\beta} = \dfrac{a}{y}$; whence

$$\frac{dx}{x} = -\frac{dy}{y}.$$

$$\therefore \int_a^\beta \varphi \left[\frac{x}{\beta} \right] \frac{dx}{x} = -\int_\beta^a \varphi \left[\frac{a}{y} \right] \frac{dy}{y} = \int_a^\beta \varphi \left[\frac{a}{x} \right] \frac{dx}{x}.$$

Now

$$I \equiv \int_a^\beta \log \frac{\beta - x}{x - a} \frac{dx}{x} = \int_a^\beta \log \left\{ \frac{\beta \left[1 - \dfrac{x}{\beta} \right]}{x \left[1 - \dfrac{a}{x} \right]} \right\} \frac{dx}{x};$$

whence

$$I = \int_a^\beta \log \frac{\beta}{x} \frac{dx}{x} + \int_a^\beta \log \left[1 - \frac{x}{\beta} \right] \frac{dx}{x} - \int_a^\beta \log \left[1 - \frac{a}{x} \right] \frac{dx}{x}.$$

By the above lemma, the last two integrals are equal. We have, also,

$$\frac{dx}{x} = -d \log \frac{\beta}{x};$$

whence

$$I = -\int_a^\beta \log \frac{\beta}{x} \, d \left[\log \frac{\beta}{x} \right] = \tfrac{1}{2} \left[\log^2 \frac{\beta}{x} \right]_\beta^a,$$

* The index, 2, is omitted in the text; it is undoubtedly an error on the part of the printer.

and

$$\int_{\alpha}^{\beta} \log \frac{\beta - x}{x - \alpha} \frac{dx}{x} = \tfrac{1}{2} \left[\log \frac{\beta}{\alpha} \right]^2.$$

[*H. L. Rice.*]

380

THE sides of a variable rectangle pass through four fixed points. Find the position of the rectangle and its dimensions when its area is a maximum.

[*Geo. R. Dean.*]

SOLUTION.

Let A, B, C, D be the four points in order; a, b the diagonals AC, BD; and α the angle between them. Then if φ be inclination of a side p to AC, the two sides are

$$p = a \cos \varphi , \qquad q = b \sin (\varphi + \alpha) ;$$

and the area

$$pq = ab \sin (\varphi + \alpha) \cos \varphi = \tfrac{1}{2}\, ab \left[\sin (2\varphi + \alpha) + \sin \alpha\right]$$

is a maximum when $\varphi = 45° - \frac{1}{2}\alpha$. The sides are

$$\tfrac{1}{2}\, a\sqrt{2}\, (\sin \tfrac{1}{2}\alpha + \cos \tfrac{1}{2}\alpha) , \quad \tfrac{1}{2}\, b\sqrt{2}\, (\sin \tfrac{1}{2}\alpha + \cos \tfrac{1}{2}\alpha) ,$$

and are equally inclined to the bisectrices of the angles between AC and BD.

[*Wm. M. Thornton.*]

Solved also by H. L. Rice and G. B. M. Zerr.

381

FROM a point in the circumference of a circle of radius R as centre is described the external arc of a circle of radius r. Determine r so that the area of the lune shall equal that of the original circle. [*W. M. Thornton.*]

SOLUTION.

Let φ be the angle at the center of the R-circle which subtends the chord

$$r = 2R \sin \tfrac{1}{2} \varphi .$$

Equating the area of the lune to that of the R-circle, we have

$$\tfrac{1}{2}\, r^2 (\pi + \varphi) - R^2\varphi + rR \cos \tfrac{1}{2} \varphi = \pi R^2 ,$$

whence

$$\tan \varphi = \pi + \varphi .$$

From which the value of φ, determined by trial, is

$$\varphi = 77° \, 27' \, 12''.08$$

Whence

$$r = 1.25122\, R .$$

[*W. H. Echols.*]

Solved also by G. B. M. Zerr and H. L. Rice.

382

Four equal circles tangent to each other cut off equal areas from a given circle. Required the radii of the cutting circles when the aggregate area cut off from the given circle is the greatest possible. [*Artemas Martin.*]

SOLUTION.

The distance between the centres of the fixed circle and cutting circle $= r\sqrt{2}$, where r is the radius of a cutting circle. Let $R =$ radius of given circle. Then

$$4r^2 \cos^{-1}\left[\frac{3r^2 - R^2}{2\sqrt{2}\,r^2}\right] + 4R^2 \cos^{-1}\left[\frac{R^2 + r^2}{2\sqrt{2}\,Rr}\right] - 2\sqrt{6R^2r^2 - R^4 - r^4}$$

is the area cut off by the four circles. Differentiating and reducing, we get

$$r^2 \cos^{-1}\left[\frac{3r^2 - R^2}{2\sqrt{2}r^2}\right] = \tfrac{1}{2}\sqrt{6R^2r^2 - R^4 - r^4}\,. \tag{1}$$

Let

$$\cos\theta = \frac{3r^2 - R^2}{2\sqrt{2}r^2}\,. \tag{2}$$

Whence (1) becomes $\theta = \sqrt{2}\sin\theta$, the solution of which gives $\theta = 79° \, 43' \, 46''$. From (2),

$$r = .633\ R\,.$$

(1) is also satisfied by $r = (\sqrt{2} + 1)\,R$, and $r = (\sqrt{2} - 1)\,R$. The first of these values gives the four tangent circles circumscribed to the given circle, the second, the four tangent circles inscribed in the given circle.

[*G. B. M. Zerr.*]

383

If c', c'', c''' be the sides of any triangle inscribed in an ellipse, and b', b'', b''' the semi-diameters parallel to the sides, show that the area is

$$A = abc'c''c'''/(4b'b''b''')\,. \quad [\textit{W. O. Whitescarver.}]$$

SOLUTION I.

It is shown by Salmon (Conic Sections, p. 220) that if the triangle be given by the eccentric angles α, β, γ, its area is

$$= 2ab\sin\tfrac{1}{2}(\alpha - \beta)\sin\tfrac{1}{2}(\beta - \gamma)\sin\tfrac{1}{2}(\gamma - \alpha)\,.$$

He also shows on page 219 that

$$c' = 2b'\sin\tfrac{1}{2}(\alpha - \beta)\,, \quad c'' = 2b''\sin\tfrac{1}{2}(\beta - \gamma)\,, \quad c''' = 2b'''\sin\tfrac{1}{2}(\gamma - \alpha)\,;$$

whence by simple substitution we have the result above.

SOLUTION II.

The radius of the circumscribing circle is $\dfrac{b'b''b'''}{ab}$ (Salmon, p. 220); whence, at once,

$$A = \frac{abc'c''c'''}{4b'b''b'''}. \qquad \text{[}W.\ O.\ Whitescarver.\text{]}$$

Solved also by G. B. M. Zerr, H. L. Rice, and F. G. Radelfinger.

384

If c be a chord of an ellipse through the points whose eccentric angles are α and β, and b' the semi-diameter parallel to the chord, show that the area of the triangle formed by the chord and the tangents at its extremities is

$$S = \frac{abc^2}{4b'^2} \tan \tfrac{1}{2}(\alpha - \beta). \qquad \text{[}W.\ O.\ Whitescarver.\text{]}$$

SOLUTION.

The equations of the two tangents are

$$\frac{x \cos \alpha}{a} + \frac{y \sin \alpha}{b} = 1,$$

$$\frac{x \cos \beta}{a} + \frac{y \sin \beta}{b} = 1.$$

Their intersection has for its coordinates

$$x = \frac{a \cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}, \qquad y = \frac{b \sin \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}.$$

$$\therefore\ \frac{2S}{ab} = \begin{vmatrix} \cos \alpha & \sin \alpha & 1 \\ \cos \beta & \sin \beta & 1 \\ \dfrac{\cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)} & \dfrac{\sin \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)} & 1 \end{vmatrix},$$

or

$$\frac{S}{ab} = -\sin \tfrac{1}{2}(\alpha - \beta) \cos \tfrac{1}{2}(\alpha - \beta) + \cos^2 \tfrac{1}{2}(\alpha + \beta) \tan \tfrac{1}{2}(\alpha - \beta)$$
$$+ \sin^2 \tfrac{1}{2}(\alpha + \beta) \tan \tfrac{1}{2}(\alpha - \beta).$$

$$\therefore\ S = ab \tan \tfrac{1}{2}(\alpha - \beta) \sin^2 \tfrac{1}{2}(\alpha - \beta),$$

or, by Exercise 383,

$$S = \frac{abc^2}{4b'^2} \tan \tfrac{1}{2}(\alpha - \beta). \qquad \text{[}H.\ L.\ Rice.\text{]}$$

Solved also by W. O. Whitescarver, F. G. Radelfinger, and G. B. M. Zerr.

385

IF in exercise 384 a tangent be drawn parallel to the chord, show that the base of the triangle formed will be $c \sec \frac{1}{2}(\alpha - \beta)$, and its area $ab \tan^3 \frac{1}{2}(\alpha - \beta)$. [*W. O. Whitescarver.*]

SOLUTION.

If the base be t, we see that it touches the ellipse at the point $\gamma = \frac{1}{2}(\alpha + \beta)$ (see Salmon, Conic Sections, p. 219). Therefore $\varDelta$ the area of triangle is, from Exercise 373, by substituting for γ,

$$\varDelta = ab \tan \tfrac{1}{2}(\alpha - \beta) \tan^2 \tfrac{1}{4}(\alpha - \beta) \qquad (1)^*$$

or

$$= 2ab \frac{\tan^3 \frac{1}{4}(\alpha - \beta)}{1 - \tan^2 \frac{1}{4}(\alpha - \beta)}.$$

Since this triangle and that of Ex. 384 are similar we have $\frac{\varDelta}{S} = \frac{t^2}{c^2}$, which gives

$$t = 2b' \tan \tfrac{1}{4}(\alpha - \beta). \qquad (2)^*$$

[*W. O. Whitescarver.*]

Solved abso by H. L. Rice and G. B. M. Zerr.

392

A SYSTEM of great circles intersect upon the equator of a sphere; a curve is drawn connecting points on the spherical surface where the circles of this system make a constant angle α with the meridians. Show that the stereographic projection of any such curve is a circular cubic whose equation may be written

$$\tan \alpha \tan \theta = \frac{c^2 + r^2}{c^2 - r^2},$$

c being the radius of the sphere, and one of the poles being the centre of the projection. [*R. A. Harris.*]

SOLUTION.

In stereographic projection the distance of any point from the origin is equal to the radius of the sphere multiplied by the tangent of one-half the polar distance, or $r = c \tan \frac{1}{2}\varphi$; whence

$$\sec \varphi = \frac{r^2 + c^2}{r^2 - c^2}.$$

If α is the constant angle, and θ the complement of the angle between the given meridian and the meridian of the moving point, we have

$$\sec \varphi = \tan \alpha \tan \theta.$$

[*Geo. R. Dean.*]

* The values given in the problem are wrong. The error came by mistaking $\frac{1}{4}$ for $\frac{1}{2}$ in equation (1). This of course changed (2).